JOURNAL OF COMBINATORIAL THEORY, Series B **66**, 24–33 (1996)
ARTICLE NO. 0003

Directed Cycles with Two Chords and Strong Spanning Directed Subgraphs with Few Arcs

CARSTEN THOMASSEN

Mathematical Institute, Technical University of Denmark, DK-2800 Lyngby, Denmark

Received September 4, 1992

metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Elsevier - Publisher Connector

spanning strong directed subgraph with less than $\frac{2}{3}m$ arcs. The constant $\frac{2}{3}$ is best possible. © 1996 Academic Press, Inc.

1. INTRODUCTION

Marcus [2] investigated for which nonnegative real numbers a, b the following holds: If D is a strongly 2-arc-connected digraph (directed graph) with n vertices and m arcs, then D has a strong spanning subdigraph with at most $am + b(n-1)$ arcs. He solved the problem except for the pairs (a, b) in the trapezoid with corners $[\frac{2}{3}, 0]$, $[\frac{7}{10}, 0]$, $[\frac{3}{10}, \frac{4}{5}]$, $[\frac{1}{3}, \frac{2}{3}]$. He showed that in order to dispose of that trapezoid it suffices to prove his conjecture that the pair $(a, b) = [\frac{1}{3}, \frac{2}{3}]$ satisfies the above assertion. He also showed that this would follow from his conjecture that every strongly 2-arc-connected digraph has a directed cycle with at least two arcs. That conjecture is also mentioned in [1, p. 32] and [3, p. 166]. We shall here prove the conjecture.

Marcus' argument [2] for the consequence that every 2-arc-connected digraph D with n vertices and m arcs has a strong spanning subdigraph with at most $\frac{1}{3}m + \frac{2}{3}(n-1)$ arcs is simple: We just contract a dicycle (with c chords, $c \geqslant 2$, and say p vertices) into a single vertex and apply the induction hypothesis to the resulting 2-arc-connected digraph, which then contains a strong spanning subdigraph D' with at most $\frac{1}{3}(m-p-c) + \frac{2}{3}(n-p)$ arcs. Adding C to D' gives a strong spanning subgraph of D with

$$\tfrac{1}{3}(m-p-c) + \tfrac{2}{3}(n-p) + p \leqslant \tfrac{1}{3}m + \tfrac{2}{3}(n-1)$$

arcs.

0095-8956/96 $12.00
Copyright © 1996 by Academic Press, Inc.
All rights of reproduction in any form reserved.

As $m \geqslant 2n$ it follows that D has a strong spanning subdigraph with at most $\frac{2}{3}(m-1)$ arcs. In the following 2-arc-connected digraph (in [2]) every spanning strong digraph has at least $\frac{2}{3}m-1$ arcs: For $i=1, 2, ..., k$, let $x_{1i}x_{2i}x_{3i}x_{4i}x_{1i}$ be a dicycle. Add the arcs $x_{2i}x_{4i}$ and $x_{4i}x_{2i}$ for $i=1, 2, ..., k$. Then make the following identifications: $x_{3i}=x_{1i+1}$, $i=1, 2, ..., k-1$, and $x_{3k}=x_{11}$.

2. Terminology

A *digraph D* consists of a set $V(D)$ of *vertices* and a set of ordered pairs xy of vertices called *arcs*. All arguments of this paper are valid for digraphs with parallel arcs, i.e. two or more arcs of the form xy. (We need this for the contraction argument in the Introduction.) However, for notational convenience we assume that there are no parallel arcs. If the arc xy is present, we say that x *dominates* y. We also say that xy is *incident* with x and y and that it *leaves* x and *enters* y. More generally, if $x \in A \subseteq V(D)$ and $y \in B \subseteq V(D)$ and $A \cap B = \varnothing$, then xy *leaves* A and *enters* B. The number of arcs leaving (respectively, entering) x is the *outdegree* (respectively, *indegree*) of x and is denoted $d_D^+(x)$ and $d_D^-(x)$, respectively. A *dipath* (directed path) P is a digraph with distinct vertices $x_1, x_2, ..., x_n$ and arcs x_ix_{i+1}, $1 \leqslant i \leqslant n-1$. We say that x_1 (respectively x_n) is the *first* (respectively, *last*) vertex of P and that P *starts* in x_1 and *ends* in x_n. We also say that x_2 and x_{n-1} are the *second* and *second last* vertices, respectively, and that x_i *precedes* x_j on P when $i<j$. If we add x_nx_1 to P, we obtain a *dicycle* (directed cycle). If a digraph D contains a dipath from x to y we say that y can be *reached* from x in D. D is *strong* (strongly connected) if each vertex can be reached from each other vertex. A *strong component* in a digraph is a maximal strong subdigraph. An *initial* (respectively *terminal*) component is a strong component with no arcs entering (respectively leaving) it. A *weak component* of a digraph is a connected component of the underlying undirected graph.

If H is a subdigraph of a digraph D, then $D-H$ is obtained from D by deleting all vertices and arcs of H and also all arcs incident with vertices of H. If xy is an arc, we write $D-xy$ instead of $D-\{xy\}$ and $D-x$ instead of $D-\{x\}$. A digraph D is strongly *k-arc-connected* if $D-E$ is strong for any set E of at most $k-1$ arcs.

A *chord xy* of a subdigraph H of D is an arc not in H such that $x, y \in V(H)$. The subdigraph $D(H)$ *induced* by H consists of H and all its chords.

If H, M are disjoint subdigraphs in D, then an (H, M)-*dipath* in D is a dipath from a vertex of H to a vertex of M and with all intermediate vertices in $D-(H \cup M)$. We write an (x, M)-dipath instead of an $(\{x\}, M)$-dipath.

An *admissible triple* is a triple (D, P, T), where D is a digraph, P is a dipath in D, and T is a vertex set in $D-P$. Moreover, D, P, T satisfy the following: If $S \subseteq V(D)\setminus V(P)$, $S \neq \emptyset$, then there are at least two arcs leaving S, and if, in addition $S \cap T = \emptyset$, then there are at least two arcs entering S. If we reverse the orientations of all arcs of D, then (D, P, T) is a *co-admissible* triple. The motivation for this definition is the following: If P is a dipath in a strongly 2-arc-connected digraph D, and x is a vertex in $D-P$, then we let D' be the subdigraph of D induced by the set of vertices that can be reached from x in $D-P$. We let D'' consist of $P \cup D'$ and all arcs in D from D' to P and possibly some arcs from P to D'. We let T be the set of vertices in D' which are incident with an arc not in D''. Then (D'', P, T) is an admissible triple.

The technique in the following lemma is used repeatedly in the paper. Its proof is an easy exercise.

LEMMA 2.1. *Let (D, P, T) be an admissible triple, and let P' be a dipath in D. (Possibly $P' = P$.) Let X be a vertex set that includes all vertices of $P \cup P'$, and let v be any vertex in $D-X$.*

(a) *Let A denote the set of vertices in $D-X$ that are reachable from v in $D-X$. Then every vertex in $D-A$ is reachable from $X \cup T$ in $D-A$. Let D_A denote the subdigraph of D induced by $A \cup P'$, and let T_A be a subset of A. Then (D_A, P', T_A) is an admissible triple if the following conditions hold:*

(1) *All arcs from A to X end in P';*

(2) *T_A contains $A \cap T$;*

(3) *T_A includes all vertices of A that are dominated by a vertex outside of D_A.*

Moreover, any arcs from P' to T_A can be removed and the triple remains admissible.

(b) *Let B denote the set of vertices in $D-X$ that can reach v in $D-X$. Then every vertex in $D-B$ can reach X in $D-B$. Let D_B denote the subdigraph of D induced by $B \cup P'$, and let T_B be a subset of B. Then (D_B, P', T_B) is a co-admissible triple if the following conditions hold:*

(1) *All arcs from X to B start in P';*

(2) *$B \cap T = \emptyset$;*

(3) *T_B includes all vertices of B that dominate a vertex outside of D_B.*

Moreover, any arcs from T_B to P' can be removed and the triple remains co-admissible.

3. Dicycles with Chords

Theorem 3.1. *If D is a strongly 2-arc-connected digraph, then D has a dicycle with at least two chords.*

We shall derive Theorem 3.1 from a technical extension.

Proposition 3.2. *Let (D, P_0, T) be an admissible triple with no arcs from P_0 to $D - P_0$, and let $v \in T$. Then either $D - P_0$ has a dicycle with at least two chords or else D has a dipath P_1 from T to P_0 (having only its last vertex t on P_0) such that either $P_1 - t$ has at least one chord or the second last vertex of P_1 dominates at least two vertices of P_0. Moreover, if $P_1 - t$ does not start at v, then $P_1 - t$ has at least two chords.*

Proposition 3.3. *Let (D, P_0, T) be an admissible triple and let $u \in V(D) \backslash V(P_0)$ such that u is dominated by the last vertex p of P_0. Then D has either a dicycle with at least two chords or else D has a dipath P_1 with the last vertex t (and no other vertex) on P_0 satisfying at least one of the conditions* (i), (ii), (iii) *below.*

(i) *P_1 starts in T. Either $P_1 - t$ has at least two chords or $P_1 - t$ has one chord and the second last vertex of P_1 dominates at least two vertices of P_0.*

(ii) *P_1 starts in T and contains u. Moreover, either $P_1 - t$ has at least one chord or else the second last vertex of P_1 dominates at least two vertices of P_0.*

(iii) *P_1 starts in a vertex y which is dominated by a vertex z on P_0 which precedes t (on P_0). The second last vertex of P_1 is denoted s. The sum of the number of chords of $P_1 - t$, the number of arcs $z'y$ ($z'y \neq zy$, $z'y \neq pu$) where z' is on P_0 and the number of arcs st' ($\neq st$) where t' is on P_0 is at least* 1. *This sum is at least* 2 *if P_1 does not contain u.*

First we prove Lemmas 3.4 and 3.5 below. Then we show how Theorem 3.1 follows from Proposition 3.2. Finally we prove Propositions 3.2 and 3.3 simultaneously by induction.

Lemma 3.4. *Let C be a dicycle in a digraph D such that $D - C$ is non-empty and, for each vertex set $S \subseteq D - C$, there are at least two arcs leaving S and at least two arcs entering S. Then D has a dicycle C' such that either C' has at least two chords or else $D - C'$ has a weak component which is not strong and which is contained in $D - C$.*

Proof. The proof is by induction on the number of vertices in $D - C$. If there is only one vertex x in $D - C$, then x dominates (and is dominated by) at least two vertices in C. It is then easy to find a dicycle with at least two chords incident with x. So we proceed to the induction step.

We can assume that $D-C$ is weakly connected since otherwise we apply the induction hypothesis to a weak component of $D-C$. If $D-C$ is not strong we have finished. So assume that $D-C$ is strong. Let xx' and $y'y$ be arcs of D such that x', y' are in $D-C$, and x, y are in C, and no vertex in the (x, y)-dipath on C (except x and y) dominates or is dominated by a vertex of $D-C$. (Possibly, $x=y$ or $x'=y'$ or both.) Let P be an (x', y')-dipath in $D-C$. Let C' be the dicycle in $C \cup P \cup \{xx', y'y\}$ containing P. If $C' \supseteq V(D) \backslash V(C)$, then C' has at least two chords (incident with x'). Otherwise, we apply the induction hypothesis to a weak component of $D-C'$ in $D-C$. ▮

LEMMA 3.5. *Let D, C and C' be as in Lemma 3.4. Let $A \cup B$ be a partitioning of the vertex set of a weak, non-strong component of $D-C'$ such that $A \neq \emptyset$, $B \neq \emptyset$ and no vertex of B dominates a vertex of A. Let T be the vertices of B dominated by some vertex not in B. Let S be the vertices of A dominating a vertex not in A. Let D_A (respectively D_B) be the digraph induced by $A \cup C'$ (respectively $B \cup C'$). Let P' be obtained from C' by deleting an arc. Then (D_B, P', T) is an admissible triple, and (D_A, P', S) is a co-admissible triple.*

The proof is straightforward and is left for the reader.

We now derive Theorem 3.1 from Proposition 3.2.

Proof of Theorem 3.1. Let C be any dicycle in D. If $D-C$ is empty, then clearly D has at least two chords. So assume that $D-C \neq \emptyset$. Let C' be as in Lemma 3.4. Let A, B, S, T, D_A and D_B be as in Lemma 3.5. As A and B are in the same weak component of $D-C'$, there is an arc uv from A to B. By Proposition 3.2, D_B has a dipath P_B such that either P_B is a dipath from T to C' with at least two chords or a (v, C')-dipath with at least one chord or with the property that the second last vertex of P_B dominates at least two vertices of C'. As P_B can be extended to a dicycle, we can assume that P_B has only one chord. That is, P_B starts at v. Similarly, we can assume that D_A has a (C', u)-dipath P_A with at least one chord or with the property that its second vertex is dominated by at least two vertices of C'. Now, the subdigraph of D induced by $P_A \cup P_B \cup C'$ has a dicycle with at least two chords. ▮

We now prove Proposition 3.2 and 3.3 by induction on $D-P_0$. If $D-P_0$ has only one vertex, it is easy to verify Proposition 3.2 and 3.3. So we proceed to the induction step.

Proof of Proposition 3.2. Let $Q_1: v_1 v_2 \cdots v_k v_{k+1}$ be a longest (v, P_0)-dipath. Let u be a vertex in $D-v_{k+1}$ dominated by v_k. If $u \in V(P_0) \cup V(Q_1)$, we have finished. So assume that $u \notin V(P_0) \cup V(Q_1)$. Let R be the set of vertices which can be reached from u in $D-(P_0 \cup Q_1)$. By the maximality of Q_1, no vertex of R dominates a vertex of P_0. Let P'_0 be the minimal dipath in Q_1 containing v_k and all vertices of Q_1 which are dominated by some vertex of R. Let T' be the union of $T \cap R$ and the set of vertices of R dominated by some vertex in $D-(R \cup P'_0)$. Let D' be the digraph induced by $R \cup P'_0$. Then (D', P'_0, T') is an admissible triple. We now apply (the induction hypothesis of) Proposition 3.3 to that triple. Let P'_1 be such that one of (i), (ii), (iii) of Proposition 3.3 is satisfied (with D', P'_0, T', P'_1 instead of D, P_0, T, P_1 respectively). If (iii) is satisfied, then we let P_1 be obtained from Q_1 by replacing a dipath in Q_1 by P'_1 and an arc from Q_1 to the first vertex of P'_1. If (i) or (ii) is satisfied, then P'_1 together with the dipath in Q_1 from the last vertex of P'_1 to v_{k+1} has at least two chords. (If the second last vertex of P'_1 dominates two vertices of Q_1 we can choose either of these to be the last vertex of P'_1.) So if the first vertex w of P'_1 is in T, we have finished. The same holds if w is dominated by some vertex of $Q_1 - P'_0$ or if $D-(P_0 \cup Q_1)$ has a dipath P'_2 from a vertex in T to w such that $P'_2 \cap R = \{w\}$. So assume that none of these cases occur. Now w is dominated by some vertex w' not in $Q_1 \cup R$. (By the assumption of Proposition 3.2, $w' \notin V(P_0)$.)

Let M be an initial component of $D-(Q_1 \cup P_0)$ such that $D-(Q_1 \cup P_0)$ has a dipath P_3 from a vertex z in M to w' with no intermediate vertex of P_3 in M. (Also, no intermediate vertex of P_3 or of M is in R by the definition of R. Possibly $z = w'$.) By the nonexistence of P'_2 above, M contains no vertex of T. Let P_4 be the minimal dipath of Q_1 containing v_k and all vertices of Q_1 that dominate some vertex of M. We can assume that P_4 is contained in P'_0 since otherwise $P_3 \cup P'_1$ is contained in a (v, P_0)-dipath having a chord. Let T'' be the vertices of M dominating some vertex not in M. Let D'' be the digraph induced by $M \cup P_4$ (except that if D'' does not include arcs from M to P_4). Then (D'', P_4, T'') is a co-admissible triple. We apply Proposition 3.2 (or more precisely, the induction hypothesis) to this co-admissible triple with z playing the role of v. Let P_5 denote the dipath (satisfying Proposition 3.2) to T'' from a vertex of P_4. If P_5 ends in z it has at least one chord or its second vertex is dominated by two vertices of P_4. Otherwise, P_5 has at least two chords. Now P_5 can be extended to either the desired dipath P_1 or a dicycle with at least two chords. (If P_5 ends at z, then we use also P_3 and P'_1. If P_5 has at least two chords, we extend it by adding any path from the end of P_5 to $P_0 \cup Q_1$.)

Proof of Proposition 3.3. Suppose first that either $u \in T$ or u is dominated by a vertex of $P_0 - p$. Then we consider the set A of vertices which can be reached from u in $D - P_0$. We let $D' = D(A) \cup P_0$ together with all arcs from $D(A)$ to P_0. Let T' consist of $T \cap A$ and all vertices of A

dominated by some vertex not in A. We apply Proposition 3.2 to D' (where u plays the role of v and T' plays the role of T). Using the dipath in the conclusion of Proposition 3.2 it is easy to find a dipath or dicycle satisfying the conclusion of Proposition 3.3. So we may assume that $u \notin T$ and that u is dominated by no vertex of $P_0 - p$. In particular, u is dominated by some vertex q in $D - P_0$.

Suppose now $D - P_0$ has no (u, q)-dipath. Let A be those vertices which can be reached from u in $D - P_0$. Let B be those vertices which can reach q in $D - P_0$. We can apply Proposition 3.2 to the admissible triple (D_A, P_0, T_A) where D_A consists of $D(A) \cup P_0$ and all arcs from A to P_0, and T_A is the union of $T \cap A$ and all vertices in A dominated by some vertex not in A. Moreover, u plays the role of v in Proposition 3.2. The dipath obtained from Proposition 3.2 is called P_A. If $B \cap T = \emptyset$, then we define analogously a co-admissible triple (D_B, P_0, T_B) and we use Proposition 3.2 where q plays the role of v to obtain a dipath, say P_B. If $B \cap T \neq \emptyset$, we let P_B denote any (T, q)-dipath in $D - P_0$. Now we use P_A or P_B (or both) to conclude the proof of Proposition 3.3. So we can assume that $D - P_0$ has a (u, q)-dipath. In particular, every vertex that can be reached in $D - P_0$ from q can also be reached from u.

Let r be the vertex on P_0 such that r is dominated by some vertex r' of $D - P_0$ which can be reached from u in $D - P_0$, and such that no vertex preceding r in P_0 is dominated by a vertex which can be reached from u in $D - P_0$. We consider first the case where $(D - P_0) - q$ has a (u, r')-dipath Q_1. Let T' be those vertices which can reach q in $D - (P_0 \cup Q_1)$ and which either belong to T or are dominated by some vertex of P_0 preceding r (on P_0). Suppose first that $T' = \emptyset$. Let A (respectively B) be those vertices in $D - (P_0 \cup Q_1)$ which can be reached from q (respectively can reach q) in $D - (P_0 \cup Q_1)$. Let D_A (respectively D_B) be a terminal (respectively initial) component in $D(A)$ (respectively $D(B)$). If $A = B$, then the subdigraph of D induced by $A \cup P'_0 \cup Q_1$ (where P'_0 is the dipath in P_0 from r to p) satisfies the assumption of Lemma 3.4 (where $P'_0 \cup Q_1 \cup \{pu, r'r\}$ plays the role of C). Now Lemmas 3.4, 3.5 imply the existence of a dicycle with at least two chords as in the proof of Theorem 3.1. A similar method applies when $A \neq B$ because in this case we can choose D_A and D_B such that $D_A \cap D_B = \emptyset$. We can therefore assume that $T' \neq \emptyset$.

Let P_2 be a (s, q)-dipath in $D - (P_0 \cup Q_1)$ where $s \in T'$. Extend Q_1 to a (u, P_0)-dipath P'_1 which is longest subject to the condition that $P'_1 \cap P_2 = \{\emptyset\}$. Let the last arc of P'_1 be denoted $w'w$. By the definition of r, w does not precede r on P_0. Possibly $w'w = r'r$. Let u' be a vertex dominated by w', $u' \neq w$. If $u' \in P_0 \cup P'_1 \cup P_2$, then $P'_1 \cup P_2 \cup \{qu\}$ satisfies the conclusion of Proposition 3.3. So assume that $u' \notin P_0 \cup P'_1 \cup P_2$. We let D' be the subdigraph of D induced by $P'_1 \cup P_2$ and the set of vertices which can be reached in $D - (P_0 \cup P'_1 \cup P_2)$ from u'. We let T'' be the vertices in

$D' - (P_1' \cup P_2)$ which are in T or which are dominated in D by some vertex not in D'. Then we apply Proposition 3.3 to D' where $P_1' \cup P_2 \cup \{qu\}$, u', T'' play the role of P_0, u, T respectively. The dipath P_3 in D' obtained by Proposition 3.3 can then be used to obtain the desired dipath (or dicycle with two chords) in D. This is clear if P_3 satisfies (iii) in Proposition 3.3 or if P_3 starts in a vertex x in T'' which in $D - (P_0 \cup P_1' \cup P_2 \cup P_3 - x)$ can be reached from a vertex in T or a vertex which is dominated by a vertex of P_0. So let us assume that this is not the case. Let x' be a vertex not in D' dominating x. Let F be the set of vertices in $D'' = D - (P_0 \cup D')$ which can reach (in D'') the vertex x'. Then let D''' be the subdigraph of D induced by $F \cup P_1' \cup P_2$ (minus the arcs from F to $P_1' \cup P_2$) and let T''' be the vertices of F dominating (in D) some vertex not in F. Then $(D''', P_1' \cup P_2 \cup \{qu\}, T''')$ is a co-admissible triple. We apply Proposition 3.2 to that co-admissible triple (with x' playing the role of v) and obtain a dipath which (possibly together with P_3 and part of $P_1' \cup P_2$) gives the desired dipath or dicycle. This completes the proof in the case where $D - (P_0 \cup \{q\})$ has a (u, r')-dipath Q_1. So assume that no such dipath Q_1 exists.

We are left with the case where $D - (P_0 \cup \{q\})$ has no (u, r')-dipath. We partition the vertex set of $D - (P_0 \cup \{q\})$ into sets A, B such that $u \in A$, $r' \in B$, and there is no arc from A to B. We can assume that A is the set of vertices which can be reached from u in $D - (P_0 \cup \{q\})$. We add a new arc from p to q and let P_0'' denote the dipath $P_0 \cup \{q\} \cup \{pq\}$. Let D' denote $D(A) \cup P_0''$ union all arcs in D from A to P_0''. Then (D'', P_0'', T') is an admissible triple where T' is the union of $T \cap A$ and the set of vertices of A dominated by some vertex not in A. We apply Proposition 3.2 to D' where u plays the role of v. Let P_1 denote the dipath satisfying the conclusion of Proposition 3.2. Also, $D - P_0$ contains a (u, r')-dipath. That dipath contains a (q, r')-dipath P_2 which does not intersect A.

Suppose first that P_1 starts in u. If P_1 ends in q or if the second last vertex of P_1 dominates q (in which case we can assume that P_1 ends in q), then $P_1 \cup P_2$ can be extended to a dicycle with two chords (one of which is qu). So assume that P_1 ends in P_0. If $D - (P_0 \cup P_1)$ has a dipath to q from a vertex which is in T or is dominated by a vertex on P_0 which precedes the last vertex of P_1, then it is easy to complete the proof (using the assumption that q dominates u). So assume that no such dipath exists. Then we obtain a co-admissible triple as follows: Let $P_3 = P_0 \cup P_1 \cup \{pu\}$ minus the last arc of P_1. Let D'' be the subdigraph of D induced by the set A'' of vertices that can reach q in $D - P_3$ together with P_3 and the arcs to A'' from P_3. Let T'' be the set of vertices of A'' dominating (in D) some vertex not in A''. We apply Proposition 3.2 to (D'', P_3, T'') where q plays the role of v. It is easy to complete the proof except when the dipath P_4 satisfying the conclusion of Proposition 3.2 ends in q and starts in P_1. Then

$P_4 \cap P_2 = \emptyset$. But now $P_0 \cup P_1 \cup P_2 \cup P_4 \cup \{r'r, pu\}$ contains a dicycle with at least two chords (one of which is qu).

Finally, we assume that P_1 starts in a vertex u_1 in $T' \backslash \{u\}$. By the last remark of Proposition 3.2, P_1 has at least two chords. If $u_1 \in T$ we have finished. So assume that $u_1 \notin T$. Then some vertex $u_2 \notin A$ dominates u_1. If $u_2 \in V(P_0)$ it is easy to complete the proof. So assume $u_2 \in \{q\} \cup B$. We claim that q then has outdegree $\geqslant 3$ (or otherwise we can complete the proof). This is clear if $u_2 = q$ because q dominates u and the second vertex (say q') of P_2. So assume that $u_2 \in B$. By the definition of an admissible triple, $D - qq'$ has a dipath to u_2 from $T \cup P_0$. If that dipath does not contain q, then it is disjoint from P_1, and it is easy to complete the proof. So we can assume that it contains q and hence $d_D^+(q) \geqslant 3$ as claimed.

Now let Q be any (u, r')-dipath in $D - P_0$. Let Q' be a longest (u, P_0)-dipath containing Q with last arc $w'w$, say. Possibly, $w'w = r'r$. Let P_0' be the (w, p)-dipath in P_0. Then $P_0' \cup Q' \cup \{pu\}$ is a dicycle C with the chord qu. We can assume that w' dominates a vertex w'' not in C. (If $w' = q = r'$ we use here that $d_D^+(q) \geqslant 3$.)

Let F be the set of vertices that can be reached in $D - (P_0 \cup Q')$ from w''. Note that no vertex of F dominates a vertex of P_0 by the maximality of Q'. Let D' consist of $C \cup D(F)$ and all arcs between C and F. Let T' consist of $T \cap F$ and all vertices of F dominated by some vertex not in $F \cup C$. Then we apply Proposition 3.3 to the admissible triple $(D', C - w'w, T')$ where w'' now plays the role of u. Let P' be obtained by Proposition 3.3. Then P' ends in Q'. If P' satisfies (iii), then D has a dicycle with two chords. So assume that P' starts in T'. If P' starts in a vertex x which can be reached from T or a predecessor of w on P_0 by a dipath P'' which has only its last vertex in common with D', then it is easy to complete the proof. So we can assume that this is not the case. Then we let x' be a vertex not in D' dominating x, and we let F' be the vertices which can reach x' in $D - (P_0 \cup Q' \cup F)$. Let D'' consist of $P_0 \cup Q' \cup \{pu\} \cup D(F')$ and all arcs from $P_0 \cup Q'$ to F'. Let T'' consist of $F' \cap T$ together with all vertices of F' dominating some vertex not in F'. Then $(D'', P_0 \cup Q' \cup \{pu\}, T'')$ is a co-admissible triple. We apply Proposition 3.2 and obtain a dicycle or dipath which (possibly together with P', P_0, and Q') completes the proof.

Acknowledgment

Thanks are due to Daniel A. Marcus of California State Polytechnic University, Pomona, for many helpful comments.

References

1. J.-C. Bermond and C. Thomassen, Cycles in digraphs—A survey, *J. Graph Theory* **5** (1981), 1–43.
2. D. A. Marcus, Spanning subgraphs of k-connected digraphs, *J. Combin. Theory B* **30** (1981), 21–31.
3. H.-J. Voss, Cycles and bridges in graphs, Kluwer Academic, Dordrecht, 1991.